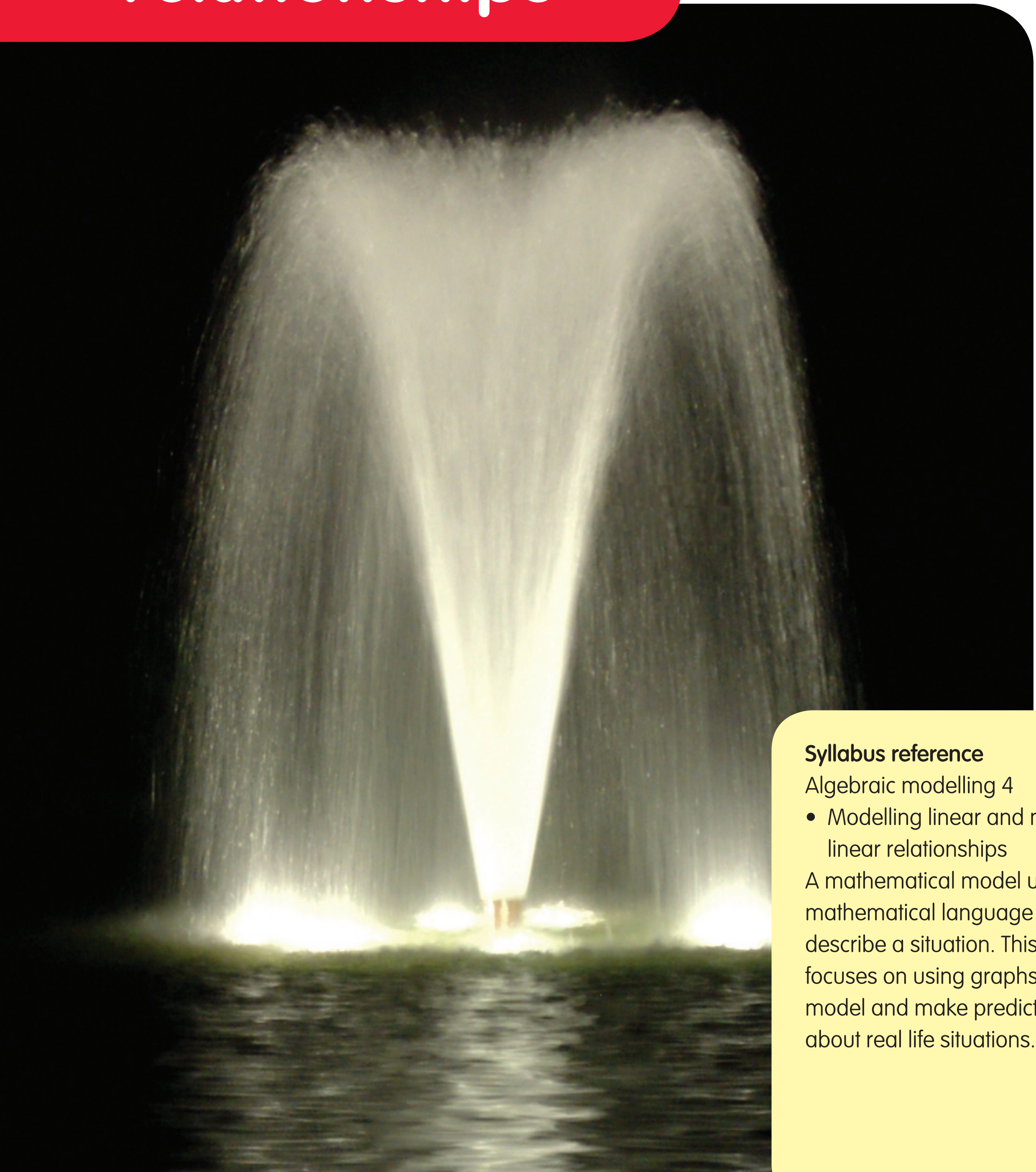
9 Modelling linear and non-linear relationships



Syllabus reference

Algebraic modelling 4

- Modelling linear and non-linear relationships

A mathematical model uses mathematical language to describe a situation. This chapter focuses on using graphs to model and make predictions about real life situations.

ARE YOU READY?

Try the questions below. If you have difficulty with any of them, extra help can be obtained by completing the matching SkillSHEET. Either click on the SkillSHEET icon next to the question on the *Maths Quest HSC Course* eBookPLUS or ask your teacher for a copy.

eBook*plus*

Digital doc
SkillSHEET 9.1
doc-1387
Substitution into a formula

Substitution into a formula

1 For each of the following linear equations, find the y-values corresponding to x, when x equals $-3, -2, -1, 0, 1, 2$ and 3. Show the results as a table of values.

a $y = 2x$ **b** $y = 3x - 1$ **c** $y = 7 - 3x$

eBook*plus*

Digital doc
SkillSHEET 9.2
doc-1388
Recognising linear functions

Recognising linear functions

2 State which of the following are linear functions.

a $y = 4x - 1$ **b** $y = x^2$ **c** $y = \dfrac{1}{x}$

d $2x - 3y + 5 = 0$ **e** $y = 2^x$ **f** $2y = 5x$

eBook*plus*

Digital doc
SkillSHEET 9.3
doc-1389
Gradient of a straight line

Gradient of a straight line

3 Calculate the gradient of the line joining the following points.

a $(1, 1)$ and $(5, 6)$ **b** $(4, 0)$ and $(6, -6)$ **c** $(-3, 7)$ and $(2, -3)$

eBook*plus*

Digital doc
SkillSHEET 9.4
doc-1390
Graphing linear equations

Graphing linear equations

4 Sketch the graph of the linear functions.

a $y = 3x$ **b** $y = 2x - 3$ **c** $y = 5 - 2x$

9A Linear functions

As discussed in chapter 5, a linear function is a function in which the highest power of the independent and dependent variables is 1. When graphed, these values form a straight line.

An example of a linear function is $y = 2x - 1$. One method of graphing the function is to create a table of values, plotting the pair of coordinates that are formed on a number plane, and joining them with a straight line. The independent variable is x, and as such, values of x are substituted into the equation to find the corresponding values of y.

If we recognise the function as linear, we need to plot only three points. Two points are sufficient to fix a line and the third is a check. If all three points are not in a straight line, we know that an error has been made.

WORKED EXAMPLE 1

Graph the equation $y = 2x - 1$.

THINK | **WRITE**

Method 1: Technology-free

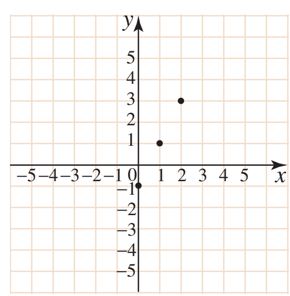
1. Draw a table of values for x. (Choose three easy values of x.)

x	0	1	2
y			

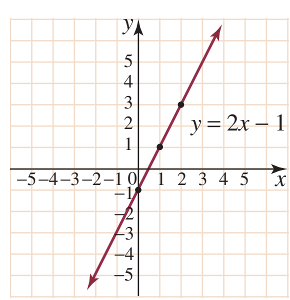
2. Substitute each value of x into the equation to find the corresponding values of y.

x	0	1	2
y	−1	1	3

3. Plot each of the points formed on a number plane.

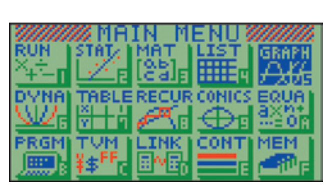


4. Join the points formed with a straight line and label the line with the equation.



Method 2: Technology-enabled

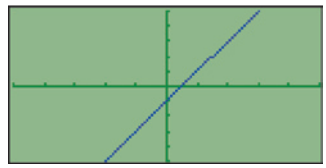
1. From the **MENU** select **GRAPH**.



2 Delete any existing equation and enter **Y1 = 2X − 1**.

3 Press SHIFT F3 **[V-Window]**. This allows you to set the lower and upper limits to draw on both the x- and y-axes. Enter the setting shown on the screen at right.

4 Press EXE to return to the previous screen, and then press F6 **(DRAW)** to draw the graph.



The straight line in worked example 1 has the equation $y = 2x - 1$, which is written in gradient–intercept form. Any equation in the form $y = mx + b$ is said to be in gradient–intercept form, because the gradient of the straight line is represented by m and the y-intercept is represented by b.

This can be used to sketch any straight line. Considering worked example 1, we can begin by plotting the point $(0, -1)$ as the y-intercept. Other points can then be plotted using the gradient, by plotting points 1 across and 2 up. That is, starting with $(0, -1)$, we plot $(1, 1)$, $(2, 3)$, $(3, 5)$ and so on.

At this point it is worth remembering the gradient formula:

$$m = \frac{\text{vertical change in position}}{\text{horizontal change in position}}$$

eBook*plus*

Interactivity
int-0804
Application of linear modelling

We use this formula when we know two points on the graph, and this is useful on many occasions to help us find the equation of a straight line.

Many real-life situations can be modelled by a linear function and/or graph. Once the equation or rule has been established, it can be used to make predictions or calculate specific values as required.

WORKED EXAMPLE 2

The Avanti car rental company charges \$80 for the hire of a car plus 22 cents per kilometre travelled.

a **How much will it cost to travel 300 kilometres?**
b **Determine the cost (\$$C$) equation for a distance of x kilometres.**
c **Graph the function for $0 \le x \le 1000$.**
d **If the final cost was \$245, what distance was covered during the hiring period?**

THINK	WRITE
a 1 Express money in the same units.	**a** 22 cents = \$0.22
2 Find the cost of travelling 300 kilometres.	Cost of travelling 300 km at \$0.22 per km: 0.22×300 $= \$66$
3 Add the charge for the hire of a car to find the total cost.	Total cost: $C = 80 + 66$ $= \$146$

b **1** Write the cost of travelling any distance x.

b Cost of travelling x km at \$0.22 per km:
$0.22 \times x$

2 Add the charge for the hire to find the total cost.

Total cost:
$C = 0.22x + 80$

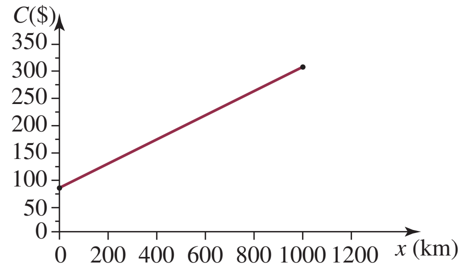
c To graph the function:

c For $0 \leq x \leq 1000$:

1 Find the coordinates of any two points. End points are usually convenient to use.

when $x = 0$, $C = 0.22 \times 0 + 80$
$= 80$
when $x = 1000$, $C = 0.22 \times 1000 + 80$
$= 300$
So the points are (0, 80) and (1000, 300).

2 Draw the set of axes with distance on the horizontal axis and cost on the vertical axis, plot the two points and join them with the straight line. Note that since neither distance nor cost can be negative, we only need the first quadrant.



d Substitute \$245 for C in the general equation of the cost and solve for x.

d $C = 0.22x + 80$
When $C = 245$:
$$245 = 0.22x + 80$$
$$165 = 0.22x$$
$$x = \frac{165}{0.22}$$
$$= 750$$
Hence, 750 kilometres were covered.

In worked example 2 both the gradient and y-intercept were known, so it was a simple matter of substituting given values into the general rule $y = mx + b$ to establish the equation of the cost. Sometimes, however, we know the gradient only, or the y intercept only, and sometimes neither of them is given. In such cases there will always be some extra information describing the relation between the variables which will enable you to find the equation.

When two linear functions are graphed on the same pair of axes, the intersection of the two graphs shows the point where both equations hold true. This can have applications in a practical context.

Graphing linear functions can be used to determine profit, loss or break-even points. If cost and receipts are graphed, the difference between the y-values at any point will determine the profit or loss. The point where the graphs intersect will be the break-even point, where no profit or loss is made.

WORKED EXAMPLE 3

The cost of producing shoes in Asia is given by the equation $C = 2000 + 15n$, where n is the number of pairs of shoes produced per day. The cost of producing shoes in Australia is given by the equation $C = 1000 + 20n$.

a **On the same pair of axes, graph the cost equations for producing shoes in Asia and Australia.**

b **When is it more cost efficient to produce the shoes in Asia?**

THINK

WRITE

Method 1: Technology-free

a 1 Draw a table of values for each cost equation.

a $C = 2000 + 15n$

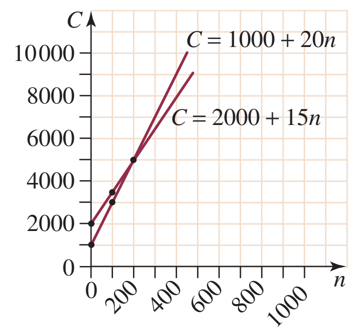
n	0	100	200
C	2000	3500	5000

$C = 1000 + 20n$

n	0	100	200
C	1000	3000	5000

2 Plot a pair of points generated by each cost equation.

3 Join each with a straight line labelling each with its equation.

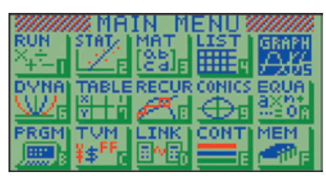


b It will be more efficient to produce the shoes in Asia after the point of intersection.

b If more than 200 pairs of shoes are produced per day, it will be cheaper to produce the shoes in Asia. This is because if $n > 200$ the value of C is less, if the shoes are produced in Asia.

Method 2: Technology-enabled

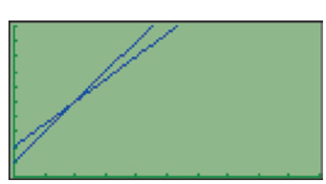
1 From the **MENU** select **GRAPH**.



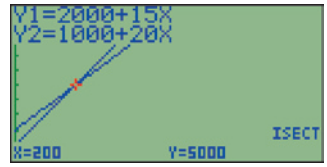
2 Delete any existing equations and enter **Y1 = 2000 + 15X and Y2 = 1000 + 20X**. Note that we replace C with **Y1** and **Y2** and n with **X**.

3 Press SHIFT F3 **[V-Window]**. This allows you to set the lower and upper limits to draw on both the x- and y-axes. Enter the setting shown on the screen at right.

4 Press EXE to return to the previous screen, and then press F6 **(DRAW)** to draw the graphs.



5 Press SHIFT F5 **[G-Solv]**, followed by F5 **[ISCT]** (intersection). This will find the point of intersection and display the coordinates of this point. Be patient: this may take a moment.



From this we can see that the intersection occurs at $x = 200$ and $y = 5000$. Interpreting this result in terms of the question shows us that when 200 pairs of shoes are produced the cost will be \$5000 in either Australia or Asia. From that point on it will be cheaper to produce the shoes in Asia.

REMEMBER

1. A linear function has powers of only 1 for both the independent and dependent variables.
2. Linear functions, when graphed, will appear as a straight line and can be written in the form $y = mx + b$, where m is the gradient and b is the y-intercept.
3. To graph a linear function, draw a table with at least three values for the independent variable and calculate the corresponding value for the dependent variable. Plot the pairs of coordinates generated and join with a straight line.
4. Linear functions can also be graphed using a graphics calculator.
5. Many practical situations can be graphed using linear functions. When two linear functions are graphed on the same pair of axes, the point of intersection will give some important information about the question.

EXERCISE 9A

Linear functions

eBook*plus*

Digital doc
SkillSHEET 9.1
doc-1387
Substitution into a formula

eBook*plus*

Digital doc
SkillSHEET 9.2
doc-1388
Recognising linear functions

eBook*plus*

Digital docs
SkillSHEET 9.3
doc-1389
Gradient of a straight line
SkillSHEET 9.4
doc-1390
Graphing linear equations

1 WE1 Graph the function $y = x + 3$.

2 Graph each of the following linear functions on separate axes.

a $y = 2x$ **b** $y = 3x - 2$
c $y = -x$ **d** $y = 5 - 2x$
e $y = \frac{1}{2}x + 3$ **f** $y = 1 - \frac{1}{4}x$

3 Consider the linear function $3x + 2y - 6 = 0$.

a Copy and complete the table at right.
b Graph the function $3x + 2y - 6 = 0$.

x	0	2	4
y			

4 WE2 The cost, C, of a taxi hire is given by the linear equation $C = 3 + 1.5d$, where d is the distance travelled in kilometres.

a Copy and complete the table below.

d	0	5	10	30
C				

b Graph the cost function for the taxi hire.
c Use the graph to determine the cost of a 20 km taxi journey.
d Katie has \$24. How far can Katie afford to travel in a taxi?

eBook*plus*

Digital doc
EXCEL Spreadsheet
doc-1391
Plotting linear graphs

5 A concert promoter finds that the profit made on a performance is given by the equation $P = 3n - 24\,000$, where n is the number of people who attend the concert.

a Complete this table of values, and use it to graph the profit equation.

n	0		10 000
p		0	

b What profit will the promoter make if 20 000 people attend the concert?

c What will be the financial outcome for the promoter if 5000 people attend the concert?

d How many people will need to attend the concert for the promoter to break even?

6 It is found that the number of ice-creams that will be sold during a day at the beach decreases as the price of the ice-creams increases. The number of ice-creams that will be sold can be determined by the equation $N = 1000 - 5P$, where P is the price of the ice-creams in cents.

a Graph the function.

b How many ice-creams will be sold at \$1 each?

c If the ice-cream salesman has only 100 ice-creams to sell, at what price should he sell them?

7 Two linear functions are represented by $y = 4 - x$ and $y = 3x$.

a Graph both linear functions on the same pair of axes.

b What is the point of intersection of the two graphs?

8 By graphing both functions on the same pair of axes, find the point of intersection of the graphs $y = 2x - 6$ and $y = x - 1$.

9 Find the point of intersection of the graphs $x + 2y - 4 = 0$ and $y = 2x + 2$.

10 WE3 A factory produces two types of computer games: game A and game B.

a The factory can produce a maximum of 120 games per week. This can be represented by the linear equation $A + B = 120$. Graph this function.

b Sales research shows that twice as many copies of game A will sell as game B. This can be represented by the equation $2A = B$. On the same pair of axes graph this function.

c Find the point of intersection of the two graphs and make a conclusion about the number of each game that should be produced by the factory each week.

11 The cost of running an old refrigerator is \$1.20 per day. This can be represented by the equation $C = 1.2d$. A new refrigerator will cost \$900 but the cost to run will be only 30c per day. This can be represented by the equation $C = 900 + 0.3d$.

a Copy and complete the table below.

d	0	1000	2000
C (old)			
C (new)			

b Graph both linear functions on the same pair of axes.

c Find the point of intersection of the two graphs; hence, state after how many days it will become more economical to purchase a new refrigerator.

12 The cost, in dollars, of producing calculators can be given by the equation $C = 15n + 1500$, where n is the number of calculators produced. When selling the calculators the receipts can be given by the equation $C = 20n$.

a Graph both linear functions on the same pair of axes.

b Determine the number of calculators that need to be sold in order for the manufacturer to break even.

Further development

13 Amex Car Rentals charges \$75 per day plus \$15 per hundred kilometres.

- **a** How much would it cost to rent a car for one day if the car travelled 265 km?
- **b** The bill for one day's rental came to \$142.50. How many kilometres did the car travel?
- **c** Sketch a graph of the cost of renting the car for one day (C) versus the number of kilometres travelled (d) .

14 An employee of a telecommunications company sells mobile-phone plans. She is offered two different salary packages by her employer:
Plan A: \$400 per week plus \$25 for each plan sold
Plan B: \$150 per week plus \$45 for each plan sold.

- **a** Copy and complete the following table.

Number of plans sold	Salary package A	Salary package B
5		
10		
15		
20		

- **b** On the same set of axes sketch the graph of salary (S) versus number of plans (n) for both Package A and Package B.
- **c** How many plans need to be sold before salary package B is the better package?

Conversion of temperature

To convert a temperature from degrees Celsius to degrees Fahrenheit, you can use the formula $F = \frac{9C}{5} + 32$. A simpler but less accurate way is to double degrees Celsius and add 30. This approximation written as a formula becomes $F = 2C + 30$.

1 Use a spreadsheet or graphics calculator to graph each function on the same set of axes.

2 Describe the accuracy of the simpler formula and state the values for which it is most accurate.

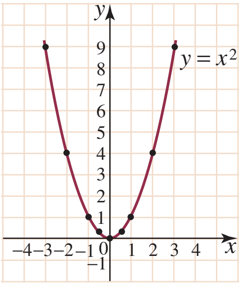
9B Quadratic functions

A **quadratic function** is a function in which the highest power of the independent variable (x) is 2. The graph of a quadratic function is a parabola, a curved line that comes to either a minimum or maximum point.

The graph of a quadratic function can be drawn by creating a table of values and plotting the pairs of coordinates generated. Because the graph is not a straight line, it is necessary to plot more than just three points to show the shape of the curve.

The most basic quadratic function is $y = x^2$. The table of values is drawn showing at least nine values of x, to ensure there one enough points to accurately demonstrate the shape.

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
y	9	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4	9



Plotting these points gives the graph shown on the right. The points are joined with a smooth curve.

This graph has a minimum at (0, 0) and forms the basic shape for all parabolas.

In general, the form of a quadratic function is $y = ax^2 + bx + c$, and for now we need consider only positive values of x.

WORKED EXAMPLE 4

Consider the quadratic function $y = x^2 - 4x + 7$.

a **Complete the table of values below.**

x	0	1	2	3	4	5
y						

b **Graph the function for $x \geq 0$.**

c **State the minimum value of $y = x^2 - 4x + 7$.**

THINK / **WRITE**

Method 1: Technology-free

a Substitute each value of x into the function.

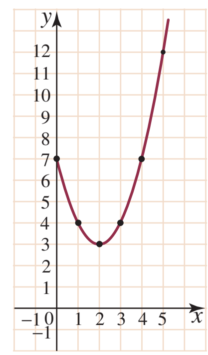
a

x	0	1	2	3	4	5
y	7	4	3	4	7	12

b 1 Plot the points generated by the table of values.

2 Join the points plotted with a smooth curve.

b

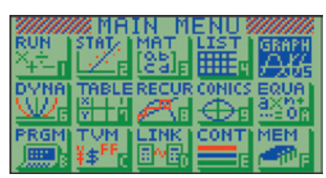


c The minimum value is the y-value at the point where the graph turns.

c For $y = x^2 - 4x + 7$, minimum value = 3.

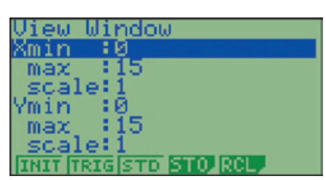
Method 2: Technology-enabled

1 From the **MENU** select **GRAPH**.

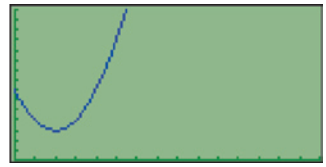


2 Delete any existing equation and enter **Y1 = X^2 − 4X + 7**.

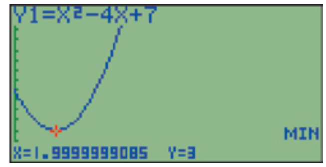
3 Press SHIFT F3 **[V-Window]**. In this course you will not need to consider negative values for x. Enter the setting shown on the screen at right.



4 Press EXE to return to the previous screen, and then press F6 **(DRAW)** to draw the graph.



5 Press SHIFT F5 **[G-Solv]**, followed by F3 **(MIN)**. This will find the minimum point and display the coordinates of that point. Be patient: this may take a moment.



Note:

1. When setting the view window you do not have to get the limit right the first time. It may take a bit of trial and error, especially with the y-values to make sure that you have the minimum (or maximum) point in your display.
2. Any question that has a negative value of x^2 (such as worked example 5) will be concave downwards and as such will have a maximum point and not a minimum point. In step 5 after pressing SHIFT F5 **[G-Solv]** you will need to press F2 **(MAX)**.
3. The display on the graphics calculator can sometimes lead to a slight inaccuracy in the answer. This can be seen in step 6. In cases such as this we can see that the calculator should display **X = 2**.

Shape of a parabola

For quadratic functions that have a positive x^2 term, the parabola is concave up. This means that the graph comes to a minimum point. When the x^2 term is negative, the graph is concave down and the graph comes to a maximum point.

WORKED EXAMPLE 5

eBook*plus*
Tutorial
int-2432
Worked example 5

Graph the function $y = 1 + 4x - x^2$.

THINK / **WRITE**

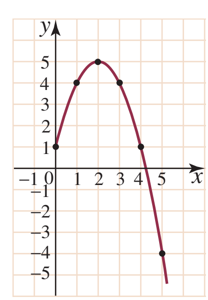
1 Draw a table of values.

2 Substitute each value of x into the function.

x	0	1	2	3	4	5
y	1	4	5	4	1	−4

3 Plot the points formed by each pair of coordinates.

4 Join the points with a smooth curve.
Note: For this function, the maximum value is 5.



Quadratic models can be used to solve several practical situations.

WORKED EXAMPLE 6

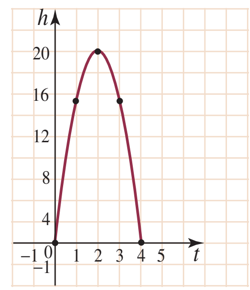
A ball is thrown in the air. Its height, h, after t seconds can be given by the formula $h = 20t - 5t^2$. Graph the function to calculate the maximum height the ball will reach.

THINK / WRITE

1. Draw a table of values.
2. Substitute the values of t to calculate the corresponding values of h.

t	0	1	2	3	4	5
h	0	15	20	15	0	–25

3. Plot the points formed by each pair of coordinates. Negative values of h can be ignored because height must be positive.
4. Join the points formed with a smooth curve.



5. The maximum height reached by the ball will be the h value at the turning point on the curve.

The maximum height reached by the ball is 20 m.

REMEMBER

1. A quadratic function is a function where the independent variable is raised to the power of 2.
2. The graph of a quadratic function is a parabola. The parabola is a curved graph and can be drawn using a table of values that has several points to allow the shape of the graph to be formed.
3. If the x^2 term is positive, the graph is concave up and has a minimum point. If the x^2 term is negative, the graph is concave down and has a maximum point.
4. The maximum or minimum value in a practical situation is the dependent variable at the maximum or minimum point.

EXERCISE 9B Quadratic functions

eBook*plus*

Digital doc
EXCEL Spreadsheet
doc-1392
Graphing quadratics

1 WE4 For the quadratic function $y = x^2 - 2x + 3$:

a copy and complete the table of values below

x	0	1	2	3	4	5
y						

b draw the graph of the function

c state the minimum value of $x^2 - 2x + 3$.

2 For the quadratic function $y = x^2 - 4x - 2$, draw up a table of values and use the table to draw the graph of the function for $x \geq 0$.

3 Graph each of the following functions for $x \geq 0$.

a $y = x^2 - 6x + 5$ **b** $y = x^2 + x + 5$ **c** $y = (x - 2)^2$

4 On the one set of axes, graph the following quadratic functions for $x \geq 0$.

a $y = x^2$ b $y = 2x^2$ c $y = \frac{1}{2}x^2$

5 On the one set of axes, graph each of the following quadratic functions for $x \geq 0$.

a $y = x^2$ b $y = x^2 + 2$ c $y = x^2 - 3$

6 Use your answers to questions 5 and 6 to answer the following.

a Describe the effect a coefficient of x^2 has on the graph of a quadratic function.

b Describe the effect adding a constant term has on the graph of a quadratic function.

7 Graph the function $y = (x - 1)^2 + 4$. Compare this with the graph of $y = x^2 - 2x + 5$. Explain why this occurs.

8 Graph the function $y = 2 + 2x - x^2$ for $x \geq 0$.

9 WE5 Graph each of the following functions for $x \geq 0$.

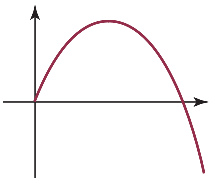
a $y = 4 + 6x - x^2$ b $y = 8 - x^2$ c $y = (2 - x)^2$

10 MC Which of the following functions is not a quadratic function?

A $y = x^2 + 5x - 4$ B $y = (x - 4)^2$

C $y = (x - 2)(x + 2)$ D $y = \dfrac{x - 2}{x + 2}$

11 MC The graph drawn on the right could have the equation:



A $y = (x - 2)^2 + 3$ B $y = (x - 2)^2 - 3$

C $y = 4 - (2 - x)^2$ D $y = (2 - x)^2 - 3$

12 MC Which of the following functions will produce the same graph as $y = (x - 4)^2 + 3$?

A $y = x^2 - 4x - 1$

B $y = x^2 - 4x + 19$

C $y = x^2 - 8x - 1$

D $y = x^2 - 8x + 19$

13 Graph the quadratic function $y = 2x^2 - 4x + 8$ for $x \geq 0$.

14 WE6 An object dropped from a height falls to Earth according to the equation $d = 5t^2$, where d is the distance fallen in metres and t is the time in seconds since the object was dropped.

a Draw the graph of d against t.

b How far will the object fall in 4 seconds?

c How long will it take for an object to fall a distance of 500 m?

15 The height of a ball in metres that is thrown vertically upwards is given by the equation $h = 30t - 5t^2$, where t is time in seconds.

a Draw the graph of h against t.

b Find the maximum height reached by the ball.

c Find the length of time taken for the ball to return to Earth.

16 A rectangular field is to be made out of 100 m of fencing. If the length of the field is x metres:

a show that the width of the field is $(50 - x)$ metres

b show that the area is given by the quadratic function $A = 50x - x^2$

c draw the graph of the function

d find the maximum area of the field and what dimension the field must be to give the maximum area.

eBook *plus*

Digital doc

WorkSHEET 9.1

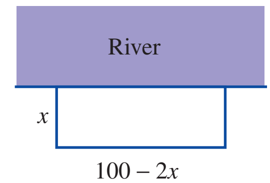
doc-1393

17 Another rectangular field is to be built with 100 m of fencing using a river as one side of the field as shown on the right.

a Show that the area of the field can be given by the equation $A = 100x - 2x^2$.

b Graph the function.

c Calculate the dimensions of the field so that the area of the field is a maximum.



Further development

18 State whether each of the following graphs is wider or narrower than the graph of $y = x^2$, and state the coordinates of the turning point of each one.

a $y = 5x^2$ **b** $y = \frac{1}{3}x^2$ **c** $y = 7x^2$ **d** $y = 10x^2$

e $\frac{2}{5}x^2$ **f** $y = 0.25x^2$ **g** $y = 1.3x^2$ **h** $y = \sqrt{3}x^2$

19 In each of the following state whether the graph is wider or narrower than $y = x^2$ and whether it has a maximum or a minimum turning point.

a $y = 3x^2$ **b** $y = {}^{-}3x^2$ **c** $y = \frac{1}{2}x^2$ **d** $y = \frac{{}^{-}1}{5}x^2$

e $y = {}^{-}4x^2$ **f** $y = 0.25x^2$ **g** $y = \sqrt{3}x^2$ **h** $y = {}^{-}0.16x^2$

i $y = \frac{4}{3}x^2$ **j** $y = {}^{-}200x^2$ **k** $y = \sqrt{5x^2}$ **l** $y = {}^{-}\sqrt{11}x^2$

20 The distance, d, of a rocket from a satellite is given by the equation $d = 5t^2 - 100t$, where t is the number of hours since the rocket was launched. At what value of t will the rocket reach the satellite?

21 Julie breeds sea monkeys. The number of sea monkeys, N, in Julie's tank is found to follow the equation $N = -0.0751\text{h}^2 + 0.69h + 200$, where h is the number of hours since the tank was supplied with food and stocked with sea monkeys.

a How many sea monkeys were there initially (i.e. at $h = 0$)

b Copy and complete the table below.

h	0	5	10	15	20	30	50
N							

c By drawing a graph estimate to the nearest hour, how long after being fed could the colony survive without further food before none were left?

22 Matilda is being pushed on a swing in her backyard. The swing follows the path given by the formula $h = \frac{1}{4}(x^2 - 3x + 4)$, where h metres is the height of the swing above the ground, x metres from the point where Matilda was pushed.

a Find the height above the ground at the point where Matilda was first pushed.

b Find the lowest distance that Matilda comes to the ground.

Maximising areas

1 Sketch ten rectangles that each have a perimeter of 40 m.

2 Show the length, width and area of each rectangle in a table.

3 If the length of the rectangle is x:
- **a** explain why the width of the rectangle will be $20 - x$
- **b** write a quadratic equation for the area of the rectangle.

4 Use a spreadsheet or graphics calculator to graph your function.

5 What is the maximum area of the rectangle?

9C Other functions

When considering modelling situations it is useful to be familiar with other non-linear graph shapes. For non straight line graphs the graphs are often curves and so several points should be found to demonstrate the shape of the curve.

Cubic functions

A **cubic function** has the independent variable (x) raised to a power of 3. Its equation is of the form $y = ax^3$

WORKED EXAMPLE 7

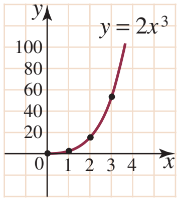
Graph the function $y = 2x^3$.

THINK

1. Draw a table of values.
2. Substitute values of x to find the corresponding values of y.
3. Plot the points generated by the table.
4. Join the points with a smooth curve.

WRITE

x	0	1	2	3
y	0	2	16	54



Hyperbolas

The equation of a **hyperbolic function** is of the form $y = \frac{a}{x}$, where a is a constant. For hyperbolas, $x \neq 0$, and so for now we graph only values of $x > 0$. As the value of x increases, the value of y will decrease, and therefore we need to look at values close to 0 when creating our table of values.

As x becomes very large, the graph approaches the x-axis but never touches it. As x becomes very small (approaches 0), the graph approaches the y-axis, but never touches it. The line $x = 0$ (the y-axis) is a vertical asymptote, and the line $y = 0$ (the x-axis) is a horizontal asymptote.
(An asymptote is a line on a graph which a curve approaches but never touches.)

WORKED EXAMPLE 8

Graph the function $y = \frac{2}{x}$.

eBook*plus*

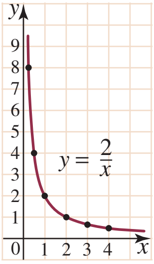
Tutorial
int-2433
Worked example 8

THINK

1. Draw a table of values.
2. Substitute the x-values into the equation to find the corresponding y-values.
3. Plot each pair of coordinates generated by the table.
4. Join each point with a smooth curve.
5. As y is never actually equal to zero the x-axis is an asymptote. As $x \neq 0$ the y-axis is an asymptote.

WRITE

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4
y	8	4	2	1	$\frac{2}{3}$	$\frac{1}{2}$



Exponential graphs

An **exponential function** is of the form $y = a^x$ or $y = b(a^x)$, where a and b are both constants. An exponential graph can increase rapidly.

WORKED EXAMPLE 9

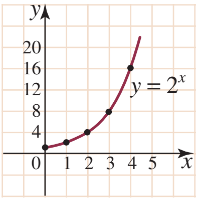
Graph the function $y = 2^x$.

THINK

1. Draw a table of values.
2. Substitute values of x to find the corresponding values of y.
3. Plot the points generated by the table.
4. Join the points with a smooth curve.
5. The x axis is an asymptote since y is never actually equal to zero.

WRITE

x	0	1	2	3	4
y	1	2	4	8	16



An exponential function of the form $y = b(a^x)$ represents an example of exponential growth. These functions show the growth of an investment over a period of time. In examples where the value of a is between 0 and 1, the function models exponential decay. An example of this is the depreciation of an asset over time.

WORKED EXAMPLE 10

Glenn invests \$10 000 at 8% p.a. with interest compounded annually. The growth of this investment can be given by the exponential function $A = 10\,000(1.08)^n$, where n is the number of years of the investment and A is the amount to which the investment grows.
Graph the growth of this investment.

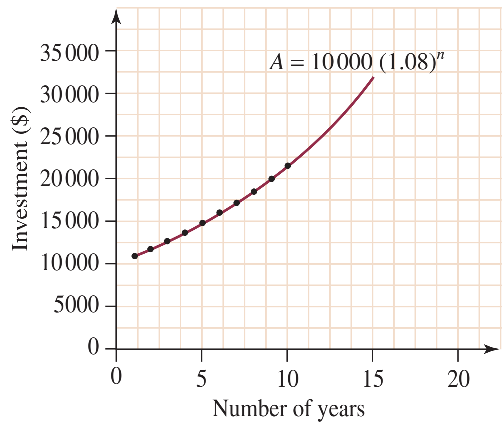
THINK

1. Draw a table of values.
2. Substitute values of n to find the corresponding values of A.

WRITE

n	1	2	3	4	5
A	10 800	11 664	12 597	13 605	14 693
n	6	7	8	9	10
A	15 869	17 138	18 509	19 990	21 589

3. Plot the points generated by the table.
4. Join the points with a smooth curve.



REMEMBER

1. A cubic equation is of the form $y = ax^3$.
2. A hyperbola is an equation of the form $y = \dfrac{a}{x}$. In such a function $x \neq 0$, and we need to examine values of x close to 0 to observe the behaviour of the curve near the y-axis.
3. An exponential function is of the form $y = b(a^x)$. An exponential function can be used to model a growth function such as the growth of an investment. If $0 < a < 1$, the function will model an exponential decay such as the depreciation of an item.

EXERCISE 9C Other functions

eBook*plus*

Digital doc
EXCEL Spreadsheet
doc-1394
Function grapher

1 WE7 Graph the cubic function $y = x^3$ for $x \geq 0$.

2 Graph the following functions for $x \geq 0$.

a $y = 3x^3$ **b** $y = \frac{1}{2}x^3$ **c** $y = -x^3$

3 Graph the hyperbolic function $y = \dfrac{4}{x}$ for $x > 0$.

4 WE8 Graph each of the following functions for $x > 0$.

a $y = \frac{1}{x}$ b $y = \frac{10}{x}$ c $y = -\frac{1}{x}$

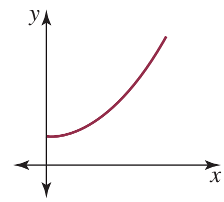
5 Graph the function $y = 3^x$.

6 WE9 Graph each of the following functions.

a $y = 4^x$ b $y = 10^x$ c $y = \left(\frac{1}{2}\right)^x$

7 Graph the function $y = 5(2^x)$.

8 MC The equation of the graph shown on the right could be:



A $y = x^3$

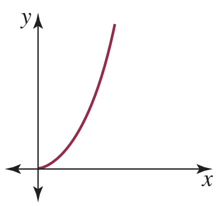
B $y = 3x$

C $y = 3^x$

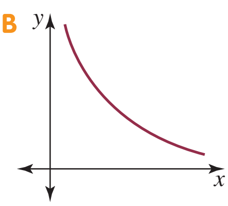
D $y = \frac{3}{x}$

9 MC Which of the graphs shown below could be the graph of $y = \frac{2}{x}$?

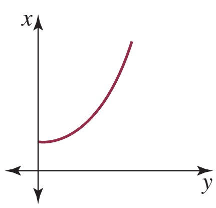
A



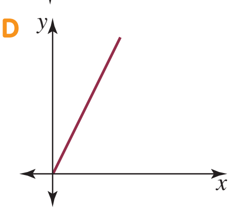
B



C



D



10 WE10 Ming Lai invests \$1000 at 10% p.a. interest with interest compounded annually. This investment can be represented by the function $A = 1000(1.1)^n$, where A is the amount to which the investment grows and n is the number of years of the investment. Draw the graph of the function.

11 Kevin invests \$50 000 at 12% p.a. interest, compounded annually.

a Write an equation for the amount, A, to which the investment will grow in terms of the number of years of the investment, n.

b Graph the function.

c Use the graph to estimate the amount of time that it will take for the investment to reach \$70 000.

12 A new car is purchased for \$40 000. The car depreciates at the rate of 15% p.a. The value, V, of the car after a number of years, n, can be given by the equation $V = 40\,000(0.85)^n$. Graph this function.

Further development

13 Sketch the graph of each of the following for $x \geq 0$.

a $y = x^3 + 4$ b $y = x^3 - 1$ c $y = 1 - 8x^3$

14 Sketch the graph of each of the following for $x \geq 0$. Show the horizontal asymptote on your sketch by drawing a broken line in red. Label this line with its equation.

a $y = \frac{1}{x} + 1$

b $y = \frac{1}{x+1}$

15 Find the equation of the asymptote and the y-intercept for each of the following. Hence, sketch the graph of each and state its domain and range.

a $y = 2^{x-1}$

b $y = 3^{x+2}$

c $y = 2^x + 3$

Compound interest

The amount to which an investment will grow under compound interest can be found using the following formula:

$$A = P(1 + r)^n$$

Consider an investment of $10 000 at an interest rate of 8% p.a.

1 If interest is compounded annually, the amount to which the investment will grow can be given by the function $A = 10\,000(1.08)^n$, where n is the number of years. Graph this function using graphing software or a graphics calculator.

2 If interest is compounded six-monthly, the function becomes $A = 10\,000(1.04)^{2n}$. On the same set of axes graph this function.

3 Write a function that will show the amount to which the investment will grow if interest is compounded quarterly, and graph this function on the same set of axes.

4 Use the graphs drawn to describe the overall effect of shortening the compounding period.

9D Variations

From our work on measurement we know that the area of a circle is given by the formula $A = \pi r^2$, where A is the area and r is the radius of the circle.

This is an example of a quantity (area) that varies in proportion with the power of another quantity (radius). This can be written as $A \propto r^2$. The symbol $\propto$ means *in proportion to*. In this example π is the constant of variation, that is, the amount by which r^2 must be multiplied to calculate the area.

An equation of the form $y = ax^2$ or $y = ax^3$ can be used to model several variations. In such cases we may need to calculate the constant of variation from some known or given information.

WORKED EXAMPLE 11

It is known that y varies directly with the cube of x. It is known that $y = 24$ when $x = 2$. Write an equation connecting the variables x and y.

THINK	WRITE
1 Write a proportion statement.	$y \propto x^3$
2 Insert a constant of variation (k) to form an equation.	$y = kx^3$
3 Substitute the known values of x and y to find the value of k.	When $x = 2$, $y = 24$. $24 = k \times 2^3$ $= 8k$ $k = 3$
4 Replace the known value of k in the equation.	$y = 3x^3$

Once we have calculated the constant of variation, we are able to calculate one quantity given the other.

WORKED EXAMPLE 12

The surface area of a cube varies directly with the square of the length of the cube's edge.

a **A cube of edge length 5.5 cm has a surface area of 181.5 cm^2. Find the constant of variation.**

b **Find the surface area of a cube with an edge length of 7.2 cm.**

THINK	WRITE
a 1 Write a proportion statement choosing pronumerals s and e.	**a** $s \propto e^2$
2 Insert the constant of variation, k, to form an equation.	$s = ke^2$
3 Substitute known information.	When $e = 5.5$, $s = 181.5$ $181.5 = k \times 5.5^2$
4 Calculate 5.5^2.	$181.5 = k \times 30.25$
5 Solve the equation (divide by 30.25).	$k = 6$
b 1 Rewrite the proportion statement with $k = 6$.	**b** $s = 6e^2$
2 Substitute $e = 7.2$.	When $e = 7.2$, $s = 6 \times 7.2^2$
3 Calculate s.	$s = 311.04$
4 Give a written answer.	The surface area of a cube with an edge of 7.2 cm is 311.04 cm^2.

WORKED EXAMPLE 13

Research conducted by a physiotherapist has determined that the height-to-mass rate (in cm/kg) of adult males is 2.26. Use this information to predict:

a **the height of a 70-kg adult male**

b **the mass of a 180-cm adult male.**

(Round answers to 1 decimal place.)

THINK	WRITE
a 1 Define the variables.	**a** Let h cm be the height of an adult male. Let w kg be the mass of an adult male.
2 Use the given rate to find the height.	$\frac{h}{w} = 2.26$ $h = 2.26w$
3 Use the rule to predict the height.	If $w = 70$, $h = 2.26 \times 70$ $= 158.2$
4 Answer the question in a sentence.	The height of an adult male with a mass of 70 kg is about 158.2 cm.
b 1 Use the rule to predict the mass.	**b** If $h = 180$, $180 = 2.26w$ $w = 79.6$
2 Answer the question in a sentence.	The mass of an adult male who is 180 cm tall is about 79.6 kg.

WORKED EXAMPLE 14

A new car has a fuel consumption of 7.2 L/100 km (this means it requires 7.2 L of petrol to travel 100 km).

a How much fuel is required for a journey of 1134 km?

b The previous model of the same car had a fuel consumption of 7.8 L/100 km. Which model is more economical to run? (Round answers to 2 decimal places.)

THINK	WRITE
a 1 Define the variables.	**a** Let L be the number of litres of fuel. Let d km be the distance travelled.
2 Find the rate relating the variables.	$\frac{L}{d}=\frac{7.2}{100}$ $\frac{L}{d}=0.072$
3 Use the rate to find the rule relating the number of litres and the distance travelled.	$L=0.072d$
4 Use the rules to find the number of litres required for a journey of 1134 km.	If $d=1134$, $L=0.072\times 1134$ $=81.648$
5 Answer the question in a sentence.	It takes about 81.65 L of fuel to travel 1134 km.
b Select the car which uses less fuel per 100 km.	**b** 7.2 L < 7.8 L The newer model is more economical to run.

WORKED EXAMPLE 15

eBook*plus*

Tutorial
int-1057
Worked example 15

If the distance, d km, travelled by a person varies directly as the time, t hours, and it is known that the person travelled 12 km while walking for 2.5 hours, find:

a how far he will travel in 3 hours

b how long he must walk in order to travel 6.72 km.

THINK	WRITE
a 1 Write the rule for k. Since $d \propto t$, then $d=kt$ and hence $k=\frac{d}{t}$.	**a** $k=\frac{d}{t}$
2 Substitute the given values for d and t into the equation and solve for k.	$k=\frac{12}{2.5}$ $=4.8$ The constant of variation is 4.8. Therefore, $\frac{d}{t}=4.8$
3 Substitute $t=3$ into the equation $\frac{d}{t}=4.8$.	When $t=3$, $\frac{d}{3}=4.8$

4 Transpose the equation to make d the subject.	$d = 4.8 \times 3$ $= 14.4$
5 Answer the question and include the appropriate unit.	He will travel 14.4 km in three hours.
b 1 Substitute $d = 6.72$ into the equation obtained in part **a**; that is, $\frac{d}{t} = 4.8$.	**b** From part **a**: $\frac{d}{t} = 4.8$ When $d = 6.72$, $\frac{6.72}{t} = 4.8$
2 Transpose the equation to make t the subject.	$6.72 = 4.8 \times t$ $4.8t = 6.72$ $t = \frac{6.72}{4.8}$ $= 1.4$
3 Convert the answer to hours and minutes by multiplying the decimal part of the answer by 60.	1.4 hours = 1 h + (0.4 × 60) min = 1 h 24 min
4 Answer the question.	In order to travel 6.72 km, he must walk for 1 h 24 min.

Although we were not actually asked to find the constant of variation, k, it was a necessary step in order to solve the problem.

REMEMBER

1. If two quantities have a constant rate or ratio, then one quantity varies directly with the other.
2. The constant rate or ratio, usually denoted k, is called the constant of variation.
3. The constant of variation k is also the gradient of the straight line graph that represents the relationship between the two quantities.
4. The constant of variation can be used to find the value of any variable given its corresponding value.

EXERCISE 9D Variations

1 WE11 It is known that y varies directly with the square of x. If $y = 88$ when $x = 4$, write an equation connecting y with x.

2 It is known that b varies directly with the cube of a. When $a = 6$, $b = 108$. Write an equation connecting b with a.

3 It is known that the distance, d, an object will fall varies directly with the square of the time, t, it has been falling. An object that has been falling for 2 seconds falls a distance of 19.6 metres.

a Write an equation connecting d with t.

b Graph the relationship between d and t.

4 WE12 The surface area of a cube varies directly with the square of its side length.

a A cube of side length 15 cm has a surface area of 1350 cm^2. Find the constant of variation.

b What is the surface area of a cube that has a side length of 6.2 cm?

5 The area of a circle varies directly with the square of its radius.

a If the area of a circle with radius 6 cm is 113.1 cm^2, find the constant of variation. (Give your answer correct to 2 decimal places.)

b What is the area of a circle with a radius of 12 cm?

6 The mass of an egg varies directly as the cube of the egg's length.

a An egg of length 5 cm has a mass of 31.25 g. Find the constant of variation.

b What will be the mass of an egg with a length of 6 cm?

c If an egg has a mass of 70 g, what would be the egg's length? (Give your answer correct to 1 decimal place.)

7 a WE13 In a study of a group of adult women, it was found that the height-to-mass rate (in cm/kg) is 2.48. Use this information to predict:

i the height of a 60 kg woman in this group. (Give your answer correct to 1 decimal place.)

ii the mass of a 170 cm woman in this group. (Give your answer correct to 1 decimal place.)

b Find the height-to-mass rate if the height is measured in metres.

8 WE14 A new sports car has a fuel consumption of 10.5 L/100 km (it requires 10.5 L of petrol to travel 100 km).

a How much fuel is required for a journey of 5430 km?

b A Nissan Pulsar has a fuel consumption of 9 L/100. Which of the cars is more economical to run?

9 In Pear Fisher Bay, land can be purchased for a price of \$5.50 per square metre.

a How much land can be purchased for \$10 000?

b What would be the cost of a block of land of 6500 square metres?

Further development

10 A large computer company can hire graduate computer programmers for a salary of \$40 000 per year, or experienced professional programmers at \$55 000 per year.

a If there is a budget of \$480 000, how many:

i graduates could be hired?

ii professionals could be hired?

b How many professional programmers are equivalent (in salary) to 23 graduate programmers?

11 An architect determines that all the windows in a new building will have a height-to-width ratio of 10 : 7 or $\frac{10}{7}$.

a Determine a rule relating height and width.

b If a window is 60 cm wide, how high is it?

c If a window is 100 cm high, how wide is it?

12 The top gear ratio on a bicycle is 7 to 2. If the larger sprocket contains 140 teeth, how many teeth does the smaller sprocket contain?

13 WE15 An experienced cyclist can travel at an average speed of 26 km/hour.

a How far can she travel in 24 hours?

b How long will it take her to travel 1000 km? (Give your answer correct to the nearest 10 minutes.)

c If she rests 1 hour after every 4 hours of travel, how long will it take her now to travel 1000 km?

14 A large four-wheel drive has a fuel consumption of 12.15 L/100 km, while for a small car it is 5.7 L/100 km.

a How many litres of fuel will be used by each vehicle for a 674 km journey? Give your answers to 1 decimal place.

b How far could the small car go on the same fuel that the four-wheel drive used to travel 1000 km? Give your answer correct to the nearest kilometre.

15 An aeroplane uses 600 L of fuel (its full tank) for a journey of 1250 km.

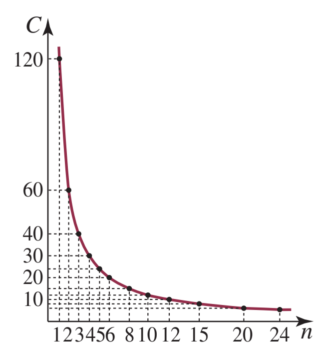
a Find the fuel consumption ratio.

b If an additional 800 L can be stored in an extra tank, what is the farthest distance that the aeroplane can travel? Give your answer correct to the nearest 10 km.

9E Inverse variation

Consider the following example. Stan used to collect basketball cards. Eventually he became bored with this hobby and decided to give all of his 120 cards to his classmates. If Stan distributed the whole collection between his 2 best friends, Mark and Eugene, they would each receive 60 cards. If he included another friend, Ashley, they would each receive 40 cards and so on. The more people who shared Stan's collection, the fewer cards each person received. There are 25 people in Stan's class, including himself. If he were to distribute 120 cards between all of his classmates, each student would receive 5 cards. This information can be represented graphically or as shown in the table.

n	1	2	3	4	5	6	8	10	12	15	20	24
C	120	60	40	30	24	20	15	12	10	8	6	5



(Note that only factors of 120 are included in order to avoid fractional answers).

It is obvious that as the number of students, n, who are to share the collection increases, the number of cards, C, that each student receives, decreases.

The product of the two variables is constant for each pair and equal to 120 — the size of the collection. That is: $1 \times 120 = 2 \times 60 = 3 \times 40 = 4 \times 30 = 5 \times 24 = 6 \times 20 = 8 \times 15 = 10 \times 12 = 120$ and so on.

Hence, the relationship between two variables can be written as:

$$C \times n = 120, \text{ or}$$

$$C = \frac{120}{n}.$$

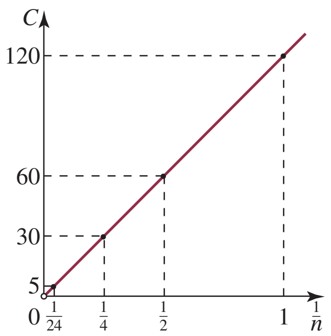
The graph of the relation is a hyperbola which has the C and n axes as its asymptotes.

Summarising our observations, we can say that the following is true for the given information:

1. An increase in one variable causes a decrease in the other.
2. The product of the two corresponding variables is a constant.
3. Neither variable is equal to 0.
4. The graph which represents the data is a hyperbola.

If we calculate the values of $\frac{1}{n}$ for each of the values in our table, we will then be able to draw a graph of C against $\frac{1}{n}$.

n	1	2	3	4	5	6	8	10	12	15	20	24
$\frac{1}{n}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{10}$	$\frac{1}{12}$	$\frac{1}{15}$	$\frac{1}{20}$	$\frac{1}{24}$
C	120	60	40	30	24	20	15	12	10	8	6	5



As you can see, the graph of C versus $\frac{1}{n}$ is a straight line directed from, but not passing through, the origin. (Note that we exclude the origin itself, hence the open circle at (0, 0), since the number of cards per person when shared between 0 students is undefined.)

Hence, we can deduce that C varies directly as $\frac{1}{n}$, that is, as the reciprocal of n.

In cases like this, we say that one variable varies inversely as (or is inversely proportional to) the other. The product of any two corresponding variables is constant and is called a constant of proportionality, k.

Hence, C is inversely proportional to n (or C varies inversely as n, or directly as the reciprocal of n). It is written as $C \propto \frac{1}{n}$.

The product of any two corresponding values of C and n is constant and equal to 120, that is $Cn = 120$. Therefore the constant of variation $k = 120$.

Therefore, the relationship between the two variables can be written as $C = \frac{120}{n}$.

Generally, for any two variables x and y, where y varies inversely as x, that is, $y \propto \frac{1}{x}$, there exists a relationship between them such that $y = \frac{k}{x}$ or $yx = k$, where k is a constant, called the *constant of proportionality* (or the *constant of variation*). The graph of the relationship is a hyperbola whereas the graph of y against $\frac{1}{x}$ is a straight line directed from, but not passing through, the origin, and having the gradient k (where $x \neq 0$). As with direct variation (Section 7A), the existence of inverse variation can be established either numerically, or graphically. Summarising this:

If $\quad y \propto \frac{1}{x}$

then $\quad y = \frac{k}{x}$

where k is the constant of variation and $k \in R\backslash\{0\}$, $x \in R\backslash\{0\}$.

WORKED EXAMPLE 16

y varies inversely with x, and $y = 10$ when $x = 2$.

a Find the constant of proportionality, k, and hence the rule relating x and y.

b Plot a graph of the relationship between x and y, for values of x which are positive factors of k less than 11.

THINK		WRITE
a	1 Write the relationship between the variables using the symbol $\propto$.	a $y \propto \frac{1}{x}$

2 Rewrite as an equation using k, the constant of variation.

$$y = \frac{k}{x}$$

3 Substitute the given values of the variables and find the value of k.

$$10 = \frac{k}{2}$$
$$k = 20$$

4 Write the rule relating the variables.

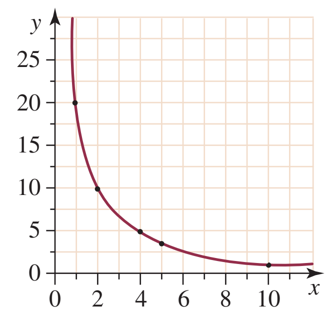
$$y = \frac{20}{x}$$

b 1 Set up a table of values for x and y, taking values for x that are positive factors of k so that only whole number values of y are obtained.

b

x	1	2	4	5	10
y	20	10	5	4	2

2 Plot the points on a clearly labelled set of axes and join the points with a smooth curve.



WORKED EXAMPLE 17

eBook*plus*

Tutorial
int-2434
Worked example 17

When a force is applied to a certain object, its acceleration varies inversely as its mass. When the acceleration of an object 12 m/s^2, the corresponding mass is 3 kg.

a Find the constant of proportionality.

b Find the rule relating acceleration and mass.

c Find the acceleration of a 1.5 kg object.

d Find the acceleration of a 6 kg object.

THINK | WRITE

a 1 Define the variables.

a Let the mass of the object be m kg.
Let the acceleration be a m/s^2.

2 Write the relationship between the variables using $\propto$.

$$a \propto \frac{1}{m}$$

3 Rewrite as an equation using k, the constant of proportionality.

$$a = \frac{k}{m}$$

4 Substitute the given values of the variables and find the value of k.

$$12 = \frac{k}{3}$$
$$k = 36$$

The constant of proportionality is 36.

b Write the rule by substituting the value of k into the equation.

b $a = \dfrac{36}{m}$

c	**1** Substitute the value of the mass into the equation to find the acceleration.	**c**	If $m = 1.5$, $a = \frac{36}{1.5}$ $= 24$
	2 Write the answer in a sentence.		The acceleration is 24 m/s^2.
d	**1** Substitute the value of the mass into the equation to find the acceleration.	**d**	If $m = 6$, $a = \frac{36}{6}$ $= 6$
	2 Write the answer in a sentence.		The acceleration is 6 m/s^2.

REMEMBER

1. For any variables x and y, where y varies inversely (or indirectly) as x, the following properties exist:
   (a) one variable increases as the other decreases
   (b) neither variable is equal to 0
   (c) the product of any pair of corresponding values is constant and equal to k
   (d) the graph which represents the relationship is a hyperbola.
2. The notation used to express y varies inversely as x is given by:

$$y \propto \frac{1}{x}$$

or $$y = \frac{k}{x}$$ where $k \in R\backslash\{0\}$ and where $x \in R\backslash\{0\}$

EXERCISE 9E Inverse variation

1 WE16 y is inversely proportional to x, and $y = 100$ when $x = 10$.
- **a** Find the constant of proportionality, k, and hence the rule relating x and y.
- **b** Plot a graph of the relationship between x and y for values of x that are positive factors of k less than 21.

2 p varies inversely as q, and $p = 12$ when $q = 4$.
- **a** Find the constant of proportionality, k, and hence the rule relating p and q.
- **b** Plot a graph of the relationship between q and p for values of q which are positive factors of k less than 11.

3 y varies inversely as x and $y = 42$ when $x = 1$.
- **a** Find the constant of proportionality, k, and hence the rule relating x and y.
- **b** Plot a graph of the relationship between x and y for values of x from 1 to 10.

4 It is known that y varies inversely with x. When $y = 10$, $x = 5$; write an equation connecting y with x.

5 It is known that m varies inversely with n. When $m = 0.5$, $n = 2$; write an equation connecting m and n.

6 The time taken, t, to travel between two points varies inversely with the average speed, s, for the trip. If the journey takes 2.5 hours at 60 km/h:
- **a** write an equation that connects t with s
- **b** graph the relationship between t and s.

7 The time, t, taken to dig a trench varies inversely with the number of workers, n, digging. It takes four workers 5 hours to dig the trench.
  a Find the constant of variation.
  b How long would it take 10 workers to dig the same trench?

8 The fuel economy, f, of a car varies inversely with the speed, s, at which it is driven. A car that averages 40 km/h has a fuel economy of 15 km/L. What will be the fuel economy of a car that averages 50 km/h?

9 In an electricity circuit, the current (measured in amps, a) is inversely proportional to the resistance (measured in ohms, r). When the resistance is 40 ohms, the current is measured at 3 amps. What will be the current when the resistance is 15 ohms?

10 WE17 When a force is applied to a certain object, its acceleration varies inversely as its mass. When the acceleration of an object is 40 m/s^2, the corresponding mass is 100 kg.
  a Find the constant of variation.
  b Find the rule relating mass and acceleration.
  c Find the acceleration of a 200-kg object.
  d Find the acceleration of a 1000-kg object.

Further development

11 The number of colouring pencils sold varies inversely with the price of each pencil. Two thousand pencils are sold at the price of $0.25 each.
  a Find the constant of proportionality.
  b Find the number of pencils that could be sold for $0.20 each.
  c Find the number of pencils that could be sold for $0.50 each.

12 The time taken to complete a journey is inversely proportional to the speed travelled. A trip is completed in 4.5 hours when travelling at 75 km per hour.
  a Find the constant of variation.
  b Find how long (to the nearest minute) the trip would take if the speed was 85 km per hour.
  c Find the speed required to complete the journey in 3.5 hours.
  d Find the distance travelled in each case.

13 The cost per person travelling in a charter plane varies inversely with the number of people in the charter group. It costs $350 per person when 50 people are travelling.
  a Find the constant of variation.
  b Find the cost per person if there are 75 people travelling.
  c Find how many people are required to reduce the cost to $250 per person.
  d Find the total cost of hiring the charter plane.

14 The electrical current in a wire varies inversely with the resistance of the wire to that current. There is a current of 10 amps when the resistance of the wire is 20 ohms.
  a Find the constant of variation.
  b Find the current when the resistance is 200 ohms.
  c Find the resistance of the wire when the current is 15 amps.

15 The time taken to complete a long computer program is inversely proportional to the speed of the computer's processor. If the program can be completed in 1 minute when the processor speed is 200 mHz, find:
  a the constant of variation
  b the time taken when the processor speed is halved
  c the processor speed required to complete the program in 20 seconds.

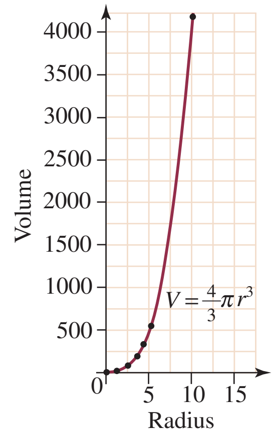
16 The time taken to complete a large building project varies inversely with the number of workers. If the building can be completed in 140 days with 75 workers, find:
  a the constant of variation
  b the time taken to complete the building with 50 workers
  c the number of workers required to complete the building within 100 days.

9F Graphing physical phenomena

In many cases, an algebraic function can be used to describe a physical situation. Consider the case of a sphere of radius r. The volume of a sphere can be given by the formula $V = \frac{4}{3}\pi r^3$. We can create a table of values that allows us to graph the function for volume.

r	0	1	2	3	4	5	10
V	0	4.19	33.51	113.10	268.08	523.60	4189

We can then plot each pair of points from the table and join the points with a smooth curve. The graph shown at right shows the relationship between volume for a sphere and radius.



WORKED EXAMPLE 18

The surface area of a sphere is given by the formula $A = 4\pi r^2$.

a Complete the table below.

r	0	1	2	3	4	5	6	7	8	9	10
A											

b Graph the surface area function.

THINK

a Substitute each value of r into the formula.

WRITE

a

r	0	1	2	3	4	5	6	7	8	9	10
A	0	12.57	50.27	113.10	201.06	314.16	452.39	615.75	804.25	1017.88	1256.64

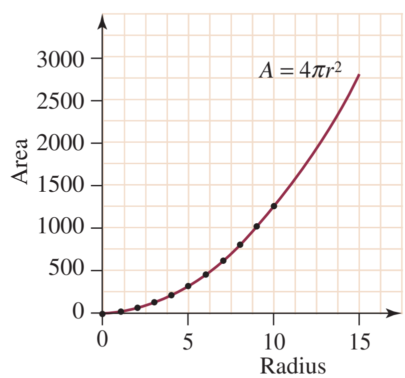
THINK

b 1 Plot each pair of points generated by the table.

2 Join the points with a smooth curve and extrapolate the graph.

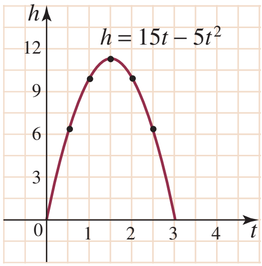
WRITE

b



Many graphs have physical restrictions placed on them. Consider the case of a ball that is thrown vertically upwards. The height, h, of the ball at any time, t, can be given by the equation $h = 15t - 5t^2$. The height of the ball must always be positive, and when the ball returns to Earth we can consider the height to be zero and so the graph stops as shown on the right.

When we graph several points, we try to estimate other values by **interpolating** (estimating values between given points by drawing the graph joining the points) or **extrapolating** (estimating values by extending the graph beyond the points given).



Other graphs need to have restrictions placed upon them when we try to interpolate or extrapolate. There may be a limit placed upon one or both of the variables, and this will indicate a change in the graph.

WORKED EXAMPLE 19

eBook*plus*

Tutorial
int-2435
Worked example 19

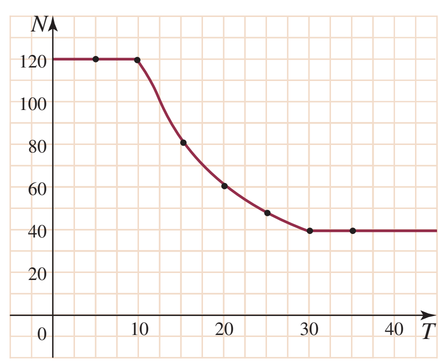
A cinema owner believes that more people will attend the movies on cold days and so believes the number of people attending each session of a movie varies inversely with the temperature of the day. When the temperature is 15°C, 80 people attend a movie. The cinema has a maximum of 120 seats, and the cinema owner believes that a minimum of 40 people will attend, regardless of temperature.

a Write an equation connecting the number of people attending the movie, N, with the temperature, T.

b Graph the relationship between attendance and temperature.

THINK	WRITE
a 1 Write an inverse proportion statement.	**a** $N \propto \dfrac{1}{T}$
2 Insert a constant of variation, k, to form an equation.	$N = \dfrac{k}{T}$
3 Substitute the known values of N and T to find the value of k.	When $T = 15$, $N = 80$ $80 = \dfrac{k}{15}$ $k = 1200$
4 Replace the known value of k in the equation.	$N = \dfrac{1200}{T}$
b 1 Create a table of values. 2 Substitute each value of T into the equation. 3 If the value of $N > 120$, then we enter 120 for N (maximum attendance); if $N < 40$, enter 40 for N (minimum attendance).	**b** (table below)
4 Plot the points and join with a smooth curve. The minimum and maximum attendance is drawn with a straight line.	(graph below)

T	5	10	15	20	25	30	35
N	120	120	80	60	48	40	40



REMEMBER

1. An algebraic model can be used to represent many physical situations.
2. When modelling a situation, there may be restrictions on one or both of the variables.

Graphing physical phenomena

1 WE18 The surface area of a cube is given by the formula $A = 6s^2$.

a Complete the table of values below.

s	0	1	2	3	4	5
A						

b Draw the graph to represent the surface area of a cube of a given side length.

2 The distance that an object will fall when dropped from a height can be given by the formula $d = 5t^2$, where d is in metres and t is in seconds. Draw a graph of the function.

3 A car is travelling at v km/h and the driver needs to brake. It takes 2.5 seconds to react and in that time the car will travel a distance of $0.7v$ m. The total stopping distance, d, can be given by the function $d = 0.01v^2 + 0.7v$.

a Copy and complete the table below.

v	0	10	20	30	40
d					

b Draw the graph of the stopping distance of a vehicle.

4 WE19 Lorraine organises a lottery syndicate at her work. If they win a prize of \$100 000, the amount is shared equally between the members of the syndicate. There must be at least one member of the syndicate and a maximum of 10.

a Write an equation putting the amount, A, each person receives in terms of the number of members, n.

b Graph the function.

5 A car is purchased new for \$40 000. After one year the depreciated value of the car is \$30 000. After the first year the car depreciates at a rate of 20% p.a.

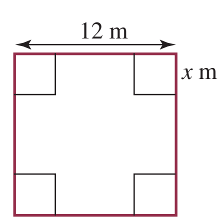
a Copy and complete the table below.

Age (years)	1	2	3	4	5
Value					

b The car will always be worth a minimum of \$2000 in scrap metal and accessories. Graph the value of the car against the age of the car.

6 The mass of a newborn baby increases by 20% per month for the first four months of life. If the average mass of a newborn baby is 3.3 kg, graph the mass function up to $n = 4$.

7 A square piece of sheet metal has a side length of 12 m. A square of side length x m is to be cut from each corner of the sheet metal and the sides bent up to form an open rectangular prism.



a What is the maximum possible value of x?

b Show that the volume of the prism formed can be given by the function $V = x(12 - 2x)^2$.

c Graph the volume function.

8 The population of a city is growing at a rate of 5% p.a. If the population in 2007 is 1.5 million, the population function can be given by the function $P = 1.5(1.05)^n$, where P is the population, in millions. The city cannot sustain a population greater than 4 000 000.

a Complete the table below.

Year	2007	2008	2009	2010	2011	2020	2027
Population (million)							

eBook*plus*

Digital doc
WorkSHEET 9.2
doc-1395

b Plot the points given and extrapolate to graph the population function.

c Use your graph to state when the population will reach its maximum sustainable level.

d What will happen to the graph when it reaches this level?

Further development

9 The following points represent a variation of the form $y = kx^2$.

x	0	2	3	4	5
y	0	12	27	48	75

Find the value of k.

10 The Safety Council conducted research on the breaking distance of vehicles and its relationship to the speed of the vehicle. The following data were obtained.

Speed (s) (km/h)	30	45	60	80	100
Breaking distance (d) (metres)	7.5	16.9	30	53.3	83.3

Find the equation relating d and s?

11 In an electrical circuit the current (I) flowing through a resistor for different resistance is shown in the table below.

Resistance (R) (ohms)	100	200	1000	1500
Current (I) (milliamps)	300	150	30	20

Deduce a relationship between I and R.

12 The data in each of the tables below exactly fit one of these rules: $y = ax^2$, $y = ax^3$, $y = \frac{a}{x}$ or $y = a\sqrt{x}$. For each set of data, plot the values of y against x, and hence select the most appropriate rule and state the value of a.

a

x	-3	-2	-1	0	1	2	3
y	-8.1	-2.4	-0.3	0	0.3	2.4	8.1

b

x	-2	-1	0	1	2	3
y	-24	-6	0	-6	-24	-54

c

x	0	0.5	1	1.5	2
y	0	1.13	1.6	1.96	2.26

d

x	1	2	4	5	10
y	5	2.5	1.25	1	0.5

e

x	-3	-2	-1	0	1	2
y	40.5	12	1.5	0	-1.5	-12

13 For her Science assignment, Rachel had to find the relationship between the intensity of the light, I, and the distance between the observer and the source of light, d. From the experiments she obtained the following results.

d	1	1.5	2	2.5	3	3.5	4
I	270	120	67.5	43.2	30	22.04	16.88

a Plot the values of I against d. What form of relationship does the graph suggest?
b Nathan (Rachel's older brother) is a Physics student. He tells Rachel that from his studies he is certain that the relationship is of the type $I = \frac{a}{d^2}$. Use this formula to help Rachel to find the model for the required relationship.

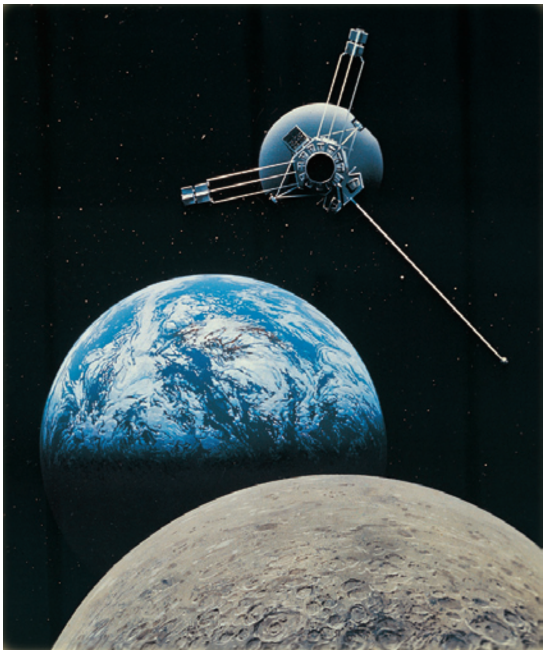
14 Joseph is a financial adviser. He is studying the prices of shares of a particular company over the last 10 months.

Months	1	2	3	4	5	6	7	8	9	10
Price ($)	6.00	6.80	7.45	8.00	8.50	8.90	9.30	9.65	10.00	10.30

a Represent the information graphically.
b Establish a suitable mathematical model, which relates the share price, P, and the number of the month, m.
c Use your model to help Joseph predict the share price for the next 2 months.

Force of gravity

When an object is dropped, the distance that it will fall in t seconds can be approximated by the formula $d = 5t^2$. The coefficient of t^2 is half the force of gravity (10 m/s^2) and so will change if an object were to be dropped on another planet. For example, on the moon this equation would become $d = 0.8t^2$.



1 Use a graphics calculator or graphing software to graph the equations for both the Earth and the moon.

2 Find out the force of gravity on other planets and compare the graphs formed with that for the Earth.

SUMMARY

Linear functions

- Linear functions have powers of only 1 for both the independent and dependent variables and are graphed as straight lines.
- To graph a linear function, a table of at least three values is drawn; the points generated are plotted on a number plane and then joined with a straight line.
- The intersection of two linear functions will give the point where both conditions hold true.

Quadratic functions

- A quadratic function is a function where the independent variable is raised to the power of 2.
- The graph of a quadratic function is a parabola, a curved graph with either a minimum (positive x^2 term) or a maximum (negative x^2 term).
- A quadratic function is graphed by plotting the points formed from a table of at least seven values.

Other functions

- A cubic function uses a power of 3 for the independent variable. It is of the form $y = ax^3$.
- A hyperbola is a function of the form $y = \frac{a}{x}$. In a hyperbolic function, as one variable increases the other decreases.
- An exponential function is of the form $y = a^x$. When $a > 1$, an exponential function models exponential growth, while if $0 < a < 1$, the function models exponential decay.
- Each of these functions is graphed by plotting points from a table of values.

Variations

- A variation occurs when one quantity changes in proportion with another.
- If one quantity varies directly with another, as one increases so does the other.
- If the quantity varies directly with the square of the other, it can be expressed as a function in the form $y = ax^2$. If it varies with the cube of another, it can be expressed in the form $y = ax^3$.
- An inverse variation occurs when one quantity decreases, while the other increases. An inverse variation can be expressed in the form $y = \frac{a}{x}$.
- The constant of variation, a, is calculated by using a known quantity of each variable. Once this has been calculated, if we know one quantity we can calculate the other.

Graphing physical phenomena

- Algebraic models can be used to represent many physical situations.
- When graphing physical phenomena, we need to consider any restrictions that may exist on one or both of the variables.

CHAPTER REVIEW

MULTIPLE CHOICE

1 Which of the following equations is *not* an example of a linear function?

A $y = 2x + 1$ **B** $y = \frac{2}{x}$

C $2y = x + 1$ **D** $x + 2y + 1 = 0$

2 Which of the following quadratic equations is equivalent to $y = (x - 3)^2 + 7$?

A $y = x^2 - 3x - 2$ **B** $y = x^2 - 3x + 16$

C $y = x^2 - 6x - 2$ **D** $y = x^2 - 6x + 16$

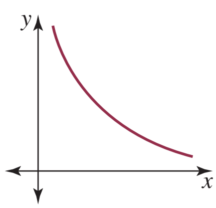
3 The graph shown below could have the equation:

A $y = x^2$

B $y = \frac{2}{x}$

C $y = 2^x$

D $y = \left(\frac{1}{2}\right)^x$



4 It is known that y varies inversely with x. The variation can be modelled by the equation:

A $y = ax$ **B** $y = ax^2$

C $y = ax^3$ **D** $y = \frac{a}{x}$

SHORT ANSWER

1 Graph each of the following linear functions.

a $y = 3x$ **b** $y = x + 3$

c $y = 2 - x$ **d** $y = 5 - 3x$

e $2y = 4x - 3$ **f** $3x - 2y + 6 = 0$

2 The cost, C, of a taxi fare is given by the formula $C = 3 + 0.4d$, where d is the distance travelled by the car, in kilometres.

a Copy and complete the table below.

d	0	5	10	15	20
C					

b Graph the cost function.

3 At a fete, 400 cans of soft drink are purchased for \$320. The cans are then sold for \$1.25 each.

a Write, as a linear function, an expression for the profit on the sale of the cans, where n is the number of cans sold.

b Graph the profit function.

c What will be the financial outcome if:

i 300 cans are sold? **ii** 142 cans are sold?

d How many cans will need to be sold for the drink stall to break even?

4 Graph the linear functions $y = 6 - x$ and $y = x + 2$, and hence state the point of intersection.

5 Andrew needs to purchase a new washing machine.

a A brand new washing machine will cost \$1000, and running costs will be approximately 20c per wash. Express this as a linear function.

b Alternatively, Andrew could purchase a second-hand washing machine for \$200, but running costs will be about \$1.00 per wash. Express this as a linear function.

c Graph both linear functions on the same pair of axes.

d By finding the point of intersection, find out after how many washes does it become more economical to purchase the new machine.

6 For the quadratic function $y = x^2 - 4x + 5$:

a copy and complete the table of values below

x	0	1	2	3	4	5
y						

b draw the graph of the function for $x \geq 0$

c state the minimum value of the function $y = x^2 - 4x + 5$.

7 For the quadratic function $y = x^2 - 2x - 2$, draw a table of values and use the table to sketch the graph for $x \geq 0$.

8 Sketch each of the following quadratic functions for $x \geq 0$.

a $y = (x - 4)^2$ **b** $y = 5 - x^2$

c $y = 4 + 2x - x^2$

9 An object is dropped from a height of 500 m. Its height above the ground at any time, t, is given by the function $h = 500 - 5t^2$.

a Draw the graph of the function.

b How many seconds does it take for the object to fall to Earth?

10 A team of workers are digging a mine shaft. The number of kilograms of earth moved each hour by the team is given by the function $E = 24n - n^2$, where n is the number of workers digging the shaft.

a Graph the function.

b What is the maximum amount of earth that can be moved by the team of workers in one hour? How many workers are needed to move this amount of earth?

c Explain possible reasons why the amount of earth moved each hour then begins to decrease as more workers are used.

11 Graph each of the following cubic functions for $x \geq 0$.

a $y = x^3$ **b** $y = \frac{1}{2}x^3$

12 Graph each of the following hyperbolic functions for $x > 0$.

a $y = \frac{1}{x}$ **b** $y = \frac{2}{x}$

13 Graph each of the following exponential functions.

a $y = 2^x$ **b** $y = (\frac{1}{2})^x$

14 The average inflation rate is 4% p.a. In 2006 it cost the average family \$500 per week in living expenses. The future cost of living, C, can be estimated using the function $C = 500(1.04)^n$ where n is the number of years since 2006.

a Graph the cost of living function.

b Use the graph to estimate the cost of living in 2016.

c When will the cost of living first reach \$1000 per week?

15 If the value of a computer purchased for \$5000 depreciates by 20% p.a., the future value of the computer, V, can be given by the equation $V = 5000(0.8)^n$, where n is the age of the computer, in years.

a Graph the function.

b Find when the value of the computer is approximately \$1000.

16 It is known that y varies directly with the square of x. When $x = 4$, $y = 80$. Write an equation connecting x with y.

17 The mass, m, of an egg varies directly with the cube of its length, l. An egg of length 5.5 cm, has a mass of 75 g.

a Write an equation connecting m with l.

b Find the mass of an egg with a length of 5 cm.

c Find the length of a 50 g egg.

18 It is known that y varies inversely with x. When $x = 8$, $y = 8$; write an equation connecting y with x.

19 The amount of food in a camp varies inversely with the number of people to feed. There is enough food to feed 100 campers for 10 days.

a Write an equation connecting the amount of food, A, with the number of campers, n.

b Calculate how long the food would last 125 campers.

c If the food lasts for four days, calculate the number of campers.

20 The area of a circle is given by the formula $A = \pi r^2$.

a Complete the table of values below.

r	0	1	2	3	4	5
A						

b Draw the graph of A against r.

21 A ball is thrown directly up in the air. The height, h, of the ball at any time, t, can be found using the equation $h = 20t - 5t^2$.

a Draw a graph of the height equation.

b Use the graph to find:

i the maximum height of the ball

ii the time taken for the ball to fall back to earth.

22 An investment of \$10 000 at 6% p.a. can be modelled using the equation $A = 10\,000(1.06)^n$, where n is the number of years of the investment.

a Graph the function.

b Use your graph to estimate the value of the investment after 8 years.

c Use your graph to find the amount of time that it will take for the investment to grow to \$15 000.

EXTENDED RESPONSE

1 As a fundraising activity, a school hires a cinema to show the premiere of a movie. The cost of hiring the cinema is \$500. People are then charged \$10 to attend the movie.

a Write a function for the profit or loss made on the movie in terms of the number of people attending.

b Graph the function.

c Use the graph to calculate the number of people who must attend the movie for the school to break even.

d A rival cinema offers to waive the hire fee but the school will receive only \$5 per person attending. On the same axes graph the function $P = 5n$.

e The school chose to pay the \$500 and receive \$10 per person. How many people must attend the premiere to make this the better of the two options?

2 A rock is thrown from a cliff 20 m above ground level. The height of the rock at any time is given by the quadratic function $h = 20 + 15t - 5t^2$.

a Copy and complete the table below.

t	0	1	2	3	4
h					

b Graph the function and use your graph to find the maximum height reached by the ball.

3 **a** On the one set of coordinate axes, sketch the graphs of $y = 2x^3$ and $y = \frac{2}{x}$.

b Use your graphs to find the point of intersection of the graphs $y = 2x^3$ and $y = \frac{2}{x}$.

4 The growth of an investment made at 8% p.a. can be modelled by the equation $y = 1.08^x$.

a Graph the function.

b Use your graph to determine the amount of time that it will take for the investment to double in value.

c The depreciation of an item at 8% p.a. can be modelled by the equation $y = 0.92^x$. Graph this function.

d Use your graph to determine the amount of time that it will take for the item to halve in value.

eBook*plus*

Digital doc
Test Yourself
doc-1396
Chapter 9

eBook*plus* ACTIVITIES

Are you ready?

Digital docs *(page 276)*

- SkillSHEET 9.1 (doc-1387): Substitution into a formula.
- SkillSHEET 9.2 (doc-1388): Recognising linear functions.
- SkillSHEET 9.3 (doc-1389): Gradient of a straight line.
- SkillSHEET 9.4 (doc-1390): Graphing linear equations.

9A Linear functions

Interactivity int-0804

- Application of linear modelling. *(page 278)*

Digital docs

- SkillSHEET 9.1 (doc-1387): Substitution into a formula. *(page 281)*
- SkillSHEET 9.2 (doc-1388): Recognising linear functions. *(page 281)*
- SkillSHEET 9.3 (doc-1389): Gradient of a straight line. *(page 281)*
- SkillSHEET 9.4 (doc-1390): Graphing linear equations. *(page 281)*
- Spreadsheet: (doc-1391): Plotting linear graphs. *(page 282)*

9B Quadratic functions

Tutorial

- WE5 int-2432: Graphing a quadratic function. *(page 285)*

Digital docs

- Spreadsheet (doc-1392): Graphing quadratics. *(page 286)*
- WorkSHEET 9.1 (doc-1393): Apply your knowledge of quadratic functions. *(page 288)*

9C Other functions

Tutorial

- WE8 int-2433: Graphing hyperbolic functions. *(page 290)*

Digital docs

- Spreadsheet (doc-1394): Function grapher. *(page 291)*

9D Variations

Tutorial

- WE15 int-1057: Calculating using variation. *(page 295)*

9E Inverse variation

Tutorial

- WE17 int-2434: Calculating using inverse variation. *(page 300)*

9F Graphing physical phenomena

Tutorial

- WE19 int-2435: Graphing a physical situation. *(page 304)*

Digital docs

- WorkSHEET 9.2 (doc-1395): Apply your knowledge of graphing physical situations. *(page 305)*

Chapter review

- Test Yourself (doc-1396): Take the end-of-chapter test to test your progress. *(page 311)*

To access eBookPLUS activities, log on to

www.jacplus.com.au